ON M-SEPARABILITY OF COUNTABLE SPACES AND FUNCTION SPACES

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ABSTRACT. We study M-separability as well as some other combinatorial versions of separability. In particular, we show that the set-theoretic hypothesis $\mathfrak{b} = \mathfrak{d}$ implies that the class of selectively separable spaces is not closed under finite products, even for the spaces of continuous functions with the topology of pointwise convergence. We also show that there exists no maximal M-separable countable space in the model of Frankiewicz, Shelah, and Zbierski in which all closed P-subspaces of ω^* admit an uncountable family of nonempty open mutually disjoint subsets. This answers several questions of Bella, Bonanzinga, Matveev, and Tkachuk.

1. INTRODUCTION

Scheepers [12] introduced a number of combinatorial properties of a topological space stronger than separability. In this paper we concentrate mainly on M-separability[1] defined as follows: a topological space X is said to be M-separable if for every sequence $\langle D_n : n \in \omega \rangle$ of dense subsets of X, one can pick finite subsets $F_n \subset D_n$ such that $\bigcup_{n \in \omega} F_n$ is dense. A topological space X is said to be *maximal* if it has no isolated points but any strictly stronger topology on X has an isolated point. The following theorems are the main results of this paper.

Theorem 1.1. *It is consistent that no countable maximal space X is M-separable.*

Theorem 1.2. ($\mathfrak{b} = \mathfrak{d}$). *There exist subspaces X_0 and X_1 of 2^ω such that $C_p(X_0)$ and $C_p(X_1)$ are M-separable, whereas $C_p(X_0) \times C_p(X_1)$ is not.*

Theorem 1.1 answers [5, Problem 3.3] in the affirmative and Theorem 1.2 shows that the negative answer to [5, Problems 3.7 and 3.9] is consistent.

Regarding Theorem 1.1, we show in section 2 that a countable maximal space which is M-separable yields a separable closed P-subset of ω^*, the remainder of the Stone-Czech compactification of ω. A model of ZFC without c.c.c. (in particular separable) closed P-subset of ω^* was constructed in [7]. We recall that a subset A of a topological space X is called a P-subset,

[1]We follow the terminology of [4].

if for every countable collection $\mathcal{U}$ of open neighborhoods of A there exists an open neighborhood V of A such that $V \subset U$ for all $U \in \mathcal{U}$.

The proof of Theorem 1.2 relies on the fact that for a metrizable separable space X, $C_p(X)$ is M-separable if and only if all finite powers of X have the Menger property (see [4, § 3] and references therein). We recall that a space X is said to have the *Menger property* if for every sequence $\langle u_n : n \in \omega \rangle$ of open covers of X there exists a sequence $\langle v_n : n \in \omega \rangle$ such that $v_n \in [u_n]^{<\omega}$ and $\cup_{n \in \omega} v_n$ is a cover of X. Assuming $\mathfrak{b} = \mathfrak{d}$, we construct in Section 3 spaces $X_0, X_1 \subset 2^\omega$ all of whose finite powers have the Menger property, whereas $X_0 \times X_1$ does not. Then the square of the disjoint union $X_0 \sqcup X_1$ does not have the Menger property (since it contains a closed copy of $X_0 \times X_1$, and the Menger property is inherited by closed subspaces), and hence $C_p(X_0 \sqcup X_1) = C_p(X_0) \times C_p(X_1)$ fails to be M-separable. At this point we would like to note that it is not even known whether there is a ZFC example of two spaces with the Menger property whose product fails to have this property (see [13, Problem 6.7]).

Under CH Theorem 1.2 can be substantially improved. Namely, by [1, Theorem 2.1] there are spaces $X, Y \subset \omega^\omega$ all finite powers of which have the Rothberger property whereas $X \times Y$ does not have the Menger property, provided that CH holds. We recall that a space X is said to have the *Rothberger property* if for every sequence $\langle u_n : n \in \omega \rangle$ of open covers of X there exists a sequence $\langle U_n : n \in \omega \rangle$ such that $U_n \in u_n$ and $\cup_{n \in \omega} U_n = X$.

While preparing this manuscript we have learned from A. Miller and B. Tsaban that CH implies the existence of γ-sets $Y_0, Y_1 \subset 2^\omega$ such that $Y_0 \times Y_1$ does not have the Menger property. It is known (see [13] and references therein) that finite powers of γ-sets are again γ-sets, and every γ-set has the Rothberger property. On the other hand, Luzin sets have the Rothberger property but they are not γ-sets. Thus this is an improvement of the result of Babinkostova [1] mentioned above.

Presently it is unknown whether the above-mentioned construction of γ-sets can be carried out under, e.g., $\omega_1 = \mathfrak{d}$. Regarding the Babinkostova result, in the Laver model we have that all sets with the Rothberger property are countable while $\mathfrak{b} = \mathfrak{d} = \mathfrak{c}$. Therefore we still believe that Theorem 1.2 can be of some interest.

In Section 4 we provide answers to a number of other questions regarding various notions of separability. These are given by citing results obtained in the framework of *selection principles in topology*, a rapidly growing area of general topology (see e.g., [13]). In this way we hope to bring more attention to this area.

In what follows, by a space we understand a metrizable separable topological space.

2. Proof of Theorem 1.1

Throughout the paper we standardly denote by

- ω^ω the space of all functions from ω to ω endowed with the Tychonov topology (here ω is equipped with the discrete topology);

- $[\omega]^{\omega}$ the set of all infinite subsets of ω;
- $[\omega]^{<\omega}$ the set of all finite subsets of ω; and
- $[\omega]^{(\omega,\omega)}$ the set $\{a \subset \omega : |a| = |\omega \setminus a| = \omega\}$ of all infinite subsets of ω with infinite complements.

A nonempty subset $\mathcal{A} \subset [\omega]^{\omega}$ is called a *semifilter* [3], if for every $A \in \mathcal{A}$ and $X \subset \omega$ such that $A \subset^* X$, $X \in \mathcal{A}$ ($A \subset^* X$ means $|A \setminus X| < \omega$). A semifilter $\mathcal{A}$ is called a *(free) filter*, if it is closed under finite intersections of its elements. Filters which are maximal with respect to the inclusion are called *ultrafilters*. We recall that a filter $\mathcal{A}$ is a called a *P-filter*, if for every sequence $\langle A_n : n \in \omega \rangle$ of elements of $\mathcal{A}$ there exists $A \in \mathcal{A}$ such that $A \subset^* A_n$ for all $n \in \omega$.

For a semifilter $\mathcal{A} \subset [\omega]^{\omega}$ we denote by $\mathcal{A}^{\perp}$ the set $\{B \in [\omega]^{\omega} : \forall A \in \mathcal{A}\, (|A \cap B| = \omega)\}$.

Now suppose that (ω, τ) is a countable maximal M-separable space. We shall construct a separable P-subset of ω^*. This suffices to prove Theorem 1.1 by the discussion following it.

Claim 2.1. *Every dense subset D of ω is open, i.e. it belongs to τ.*

Proof. Since D is dense, the topology on ω generated by $\tau \cup \{D\}$ has no isolated points. If D is not open, then this topology is strictly stronger than τ. $\qquad\square$

Claim 2.2. *Suppose that $\mathcal{F}$ and $\mathcal{A}$ are filters such that $\mathcal{A} \subset \mathcal{F}^{\perp}$. Then there exists an ultrafilter $\mathcal{U}$ such that $\mathcal{A} \subset \mathcal{U} \subset \mathcal{F}^{\perp}$.*

Proof. Let $\mathcal{U}$ be a maximal filter with respect to the property $\mathcal{A} \subset \mathcal{U} \subset \mathcal{F}^{\perp}$. We claim that $\mathcal{U}$ is an ultrafilter. If this is not true, then there exists $X \subset \omega$ such that $X, \omega \setminus X \notin \mathcal{U}$. The maximality of $\mathcal{U}$ implies that neither $\mathcal{U} \cup \{X\}$ nor $\mathcal{U} \cup \{\omega \setminus X\}$ generates a filter contained in $\mathcal{F}^{\perp}$, which means that there exist $U_0, U_1 \in \mathcal{U}$ and $F_0, F_1 \in \mathcal{F}$ such that $U_0 \cap F_0 \cap X = \emptyset$ and $U_1 \cap F_1 \cap (\omega \setminus X) = \emptyset$. It follows that $U_0 \cap F_0 \subset \omega \setminus X$ and $U_1 \cap F_1 \subset X$, and hence $(U_0 \cap U_1) \cap (F_0 \cap F_1) = \emptyset$, which contradicts the fact that $\mathcal{U} \subset \mathcal{F}^{\perp}$. $\quad\square$

Let us denote by $\mathcal{D}$ the collection of all dense subsets of (ω, τ). Claim 2.1 implies that $\mathcal{D}$ is a filter. It is easy to verify that $(\bigcup_{n\in\omega} \mathcal{A}_n)^{\perp} = \bigcap_{n\in\omega} \mathcal{A}_n^{\perp}$ for any semifilters $\mathcal{A}, \mathcal{A}_0, \mathcal{A}_1, \dots$ (see [3]).

Claim 2.3. *There exists a sequence of ultrafilters $\langle \mathcal{U}_n : n \in \omega \rangle$ such that $\mathcal{D} = \bigcap_{n\in\omega} \mathcal{U}_n$.*

Proof. Let

$$\mathcal{F}_n = \{X \cup (A \setminus \{n\}) : X \subset \omega,\ n \in A \in \tau\}.$$

It is clear that $\mathcal{F}_n$ is a filter for every n and $\mathcal{D} = (\bigcup_{n\in\omega} \mathcal{F}_n)^{\perp} = \bigcap_{n\in\omega} \mathcal{F}_n^{\perp}$. Claim 2.2 yields for every n an ultrafilter $\mathcal{U}_n$ such that $\mathcal{D} \subset \mathcal{U}_n \subset \mathcal{F}_n^{\perp}$. It follows from the above that

$$\mathcal{D} \subset \bigcap_{n\in\omega} \mathcal{U}_n \subset \bigcap_{n\in\omega} \mathcal{F}_n^{\perp} = \mathcal{D},$$

which completes the proof. $\qquad\square$

The M-separability of X simply means that $\mathcal{D}$ is a P-filter. Thus we have proved that there exists a sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of ultrafilters such that $\bigcap_{n \in \omega} \mathcal{U}_n$ is a P-filter. This obviously implies that the closure in ω^* of $\{\mathcal{U}_n : n \in \omega\}$ is a P-set, which finishes our proof.

3. Proof of Theorem 1.2

First we introduce some notations and definitions.

The Cantor space 2^ω is identified with the power-set of ω via characteristic functions. Each infinite subset a of ω can also be viewed as an element of ω^ω, namely the increasing enumeration of a. Define a preorder $\leq^*$ on ω^ω by $f \leq^* g$ if and only if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. A subset $A \subset \omega^\omega$ is called *dominating* (resp. *unbounded*), if for every $x \in \omega^\omega$ there exists $a \in A$ such that $x \leq^* a$ (resp. $a \not\leq^* x$). The minimal cardinality of an unbounded (resp. dominating) subset of ω^ω is denoted by $\mathfrak{b}$ (resp. $\mathfrak{d}$). It is a direct consequence of the definition that $\mathfrak{b} \leq \mathfrak{d}$. The strict inequality is consistent: it holds, e.g., in the Cohen model of $\neg$CH. For more information about $\mathfrak{b}, \mathfrak{d}$, and many other cardinal characteristics of this kind we refer the reader to [15].

Given a relation R on ω and $x, y \in \omega^\omega$, we denote the set $\{n \in \omega : x(n) R\, y(n)\}$ by $[x R\, y]$. For a filter $\mathcal{F}$ and elements $x, y \in \omega^\omega$ we write $x \leq_\mathcal{F} y$ if $[x \leq y] \in \mathcal{F}$. The relation $\leq_\mathcal{F}$ is easily seen to be a preorder. The minimal cardinality of an unbounded with respect to $\leq_\mathcal{F}$ subset of ω^ω is denoted by $\mathfrak{b}(\mathcal{F})$. It is easy to see that $\mathfrak{b} \leq \mathfrak{b}(\mathcal{F}) \leq \mathfrak{d}$ for any filter $\mathcal{F}$, $\leq^* = \leq_{\mathfrak{F}r}$, and hence $\mathfrak{b} = \mathfrak{b}(\mathfrak{F}r)$, where $\mathfrak{F}r$ denotes the filter of all cofinite subsets of ω.

For a filter $\mathcal{F}$, we say that $S = \{f_\alpha : \alpha < \mathfrak{b}(\mathcal{F})\}$ is a *cofinal $\mathfrak{b}(\mathcal{F})$-scale* if $f_\alpha \leq_\mathcal{F} f_\beta$ for all $\alpha \leq \beta$, and for every $g \in \omega^\omega$ there exists $\alpha < \mathfrak{b}(\mathcal{F})$ such that $g \leq_\mathcal{F} f_\alpha$. Cofinal $\mathfrak{b}(\mathfrak{F}r)$-scales are simply called *scales*. It is easy to see that for every filter $\mathcal{F}$ there exists a cofinal $\mathfrak{b}(\mathcal{F})$-scale provided $\mathfrak{b} = \mathfrak{d}$.

The following fact is a direct consequence of [14, Theorem 4.5].

Theorem 3.1. *Assume that $\mathcal{F}$ is a filter and $S = \{f_\alpha : \alpha < \mathfrak{b}(\mathcal{F})\} \subset [\omega]^\omega$ is a cofinal $\mathfrak{b}(\mathcal{F})$-scale. Then all finite powers of the set $X = S \cup [\omega]^{<\omega}$ have the Menger property.*

We shall also need the following characterization of the Menger property which is due to Hurewicz (see [11]).

Theorem 3.2. *Let X be a zero-dimensional set of reals. Then X has the Menger property if and only if no continuous image of X in ω^ω is dominating.*

A family $\mathcal{F} \subset [\omega]^\omega$ is said to be *centered* if each finite subset of $\mathcal{F}$ has an infinite intersection. Centered families generate filters by taking finite intersections and supersets. We will denote the generated filter by $\langle \mathcal{F} \rangle$. For $Y \subset \omega^\omega$, let maxfin$Y$ denote its closure under pointwise maxima of finite subsets. The proof of the following theorem is reminiscent of that of Theorem 9.1 in [14].

Theorem 3.3. ($\mathfrak{b} = \mathfrak{d}$). *There are subspaces X_0 and X_1 of 2^ω such that all finite powers of X_0 and X_1 have the Menger property, whereas $X_0 \times X_1$ does not.*

Proof. Let $\{d_\alpha : \alpha < \mathfrak{b}\} \subset [\omega]^{(\omega,\omega)}$ be a scale.

Since $P := [\omega]^{(\omega,\omega)} \cup [\omega]^{<\omega}$ is a nowhere locally compact Polish space, it is homeomorphic to $\mathbb{Z}^\omega$. Therefore there exists a map $\star : P \times P \to P$ which turns P into a Polish topological group.

For $i \in 2$, we construct by induction on $\alpha < \mathfrak{b}$ a filter $\mathcal{F}_i$ and a dominating $\mathfrak{b}(\mathcal{F}_i)$-scale $\{a_\alpha^i : \alpha < \mathfrak{b}\} \subset [\omega]^{(\omega,\omega)}$ such that $a_\alpha^0 \star a_\alpha^1 = \omega \setminus d_\alpha$. Assume that a_β^i have been defined for each $\beta < \alpha$ and $i \in 2$. Let $\mathcal{A}_\alpha^i = \text{maxfin}\{d_\beta, a_\beta^i : \beta < \alpha\}$, $\tilde{\mathcal{F}}_\alpha^i = \bigcup_{\beta<\alpha} \mathcal{F}_\beta^i$, and $\mathcal{G}_\alpha^i = \{f \circ b : f \in \mathcal{A}_\alpha^i, b \in \tilde{\mathcal{F}}_\alpha^i\}$, where $i \in 2$.

We inductively assume that $\mathcal{F}_\beta^i$, $\beta < \alpha$, is an increasing chain of filters such that $|\mathcal{F}_\beta^i| \leq |\beta|$ for each $\beta < \alpha$ and $i \in 2$. This implies that $|\mathcal{G}_\alpha^i| \leq |\alpha| < \mathfrak{b}$. Therefore there exists $c \in [\omega]^\omega$ such that $x \leq^* c$ for all $x \in \mathcal{G}_\alpha^0 \cup \mathcal{G}_\alpha^1$. Since $Y_\alpha := \{y \in [\omega]^{(\omega,\omega)} : y \not\leq^* c\}$ is a dense G_δ subset of $[\omega]^{(\omega,\omega)}$, there are $a_\alpha^0, a_\alpha^1 \in Y_\alpha$ such that $a_\alpha^0 \star a_\alpha^1 = \omega \setminus d_\alpha$. Set

$$\mathcal{F}_\alpha^i = \langle \tilde{\mathcal{F}}_\alpha^i \cup \{[f \leq a_\alpha^i] : f \in \mathcal{A}_\alpha^i\}\rangle, \ i \in 2.$$

We must show that $\mathcal{F}_\alpha^i$'s remain filters. Fix $i \in 2$. Since $\mathcal{A}_\alpha^i$ is closed under pointwise maxima, it suffices to show that $b \cap [f \leq a_\alpha^i]$ is infinite for all $b \in \tilde{\mathcal{F}}_\alpha^i$ and $f \in \mathcal{A}_\alpha^i$. Suppose, to the contrary, that $b \cap [f \leq a_\alpha^i]$ is finite. Then $a_\alpha^i \leq a_\alpha^i \circ b \leq^* f \circ b \in \mathcal{G}_\alpha^i$, which contradicts with $a_\alpha^i \not\leq^* c$ and $f \circ b \leq^* c$.

Set $X_i = \{a_\alpha^i : \alpha < \mathfrak{b}\} \cup [\omega]^{<\omega}$ and $\mathcal{F}_i = \bigcup_{\alpha<\mathfrak{b}} \mathcal{F}_\alpha^i$, $i \in 2$. By construction, $\{a_\alpha^i : \alpha < \mathfrak{b}\}$ is a cofinal $\mathfrak{b}(\mathcal{F}_i)$-scale. By Theorem 3.1, all finite powers of X_i have the Menger property. Let $\phi : 2^\omega \to 2^\omega$ be the map assigning to $x \subset \omega$ its complement $\omega \setminus x$. It follows from the above that $\{d_\alpha\}_{\alpha<\mathfrak{b}} \subset (\phi \circ \star)(X_0 \times X_1) \subset [\omega]^\omega$, and hence $X_0 \times X_1$ can be continuously mapped onto a dominating subset of $[\omega]^\omega$, which means that it does not have the Menger property. $\square$

One can also prove Theorem 3.3 by methods developed in [6] (see e.g., [2]). Moreover, one just has to "add an ϵ" to [6] to do this, and hence we believe that Theorem 3.3 might be considered as a folklore for those who had a chance to read [6].

4. Epilogue

We recall from [8] that $X \subset 2^\omega$ is called a γ-*set*, if $C_p(X)$ has the Fréchet-Urysohn property, i.e. for every $f \in C_p(X)$ and a subset $A \subset C_p(X)$ containing f in its closure, there exists a sequence of elements of A converging to f. The recent groundbreaking result of Orenstein and Tsaban [10] states that under $\mathfrak{p} = \mathfrak{b}$ there exists a γ-set of size $\mathfrak{b}$. Suppose that $\mathfrak{p} = \mathfrak{d}$, fix a γ-set $X = \{x_\alpha : \alpha < \mathfrak{d}\} \subset 2^\omega$ with x_α's mutually different, and a scale $S = \{f_\alpha : \alpha < \mathfrak{d}\} \subset \omega^\omega$. Modify S in such a way that it remains a scale and $\{n : f_\alpha(n) \text{ is even}\} = x_\alpha$. We denote the modified scale again by S. Then the γ-set X is a continuous bijective image of S, and hence $C_p(X)$ can be embedded into $C_p(S)$ as a dense subset. Thus $C_p(S)$, which

fails to be M-separable, contains a dense subset which is GN-separable by [4, Theorems 86, 57, 40] and the well known fact that all finite powers of a γ-set have the Hurewicz as well as the Rothberger properties (see [4] for all the definitions involved). Moreover, $C_p(X)$ is a dense subspace of $\mathbb{R}^{\mathfrak{d}}$, and $\{f \in C_p(X) : f(X) \subset 2\}$ is a dense subspace of $2^{\mathfrak{d}}$ which is GN-separable by [4, Proposition 90]. This implies a positive answer to [4, Questions 64, 93, and 94] under $\mathfrak{p} = \mathfrak{d}$.

By [9, Theorem 5.1], there exist a ZFC example of a space $X \subset 2^{\omega}$ of size ω_1 all of whose finite powers have the Hurewicz property. (Moreover, the space constructed in Case 2 of the proof of [9, Theorem 5.1] is a γ-set by results of [10].) Then $\{f \in C_p(X) : f(X) \subset 2\}$ is a dense hereditarily H-separable subspace of 2^{ω_1} (see [4, Theorem 40, Corollary 42]). This provides the positive answer to [5, Problem 3.1].

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